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QUANTUM OPTICS REPORT

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REPORT QUANTUM MECHANICS AND OPTICS

1. QUANTUM MECHANICS

Quantum mechanics is a theory in physics that provides a description of the behavior of matter and energy on the atomic and subatomic scale.

Comparing classical mechanics with quantum mechanics, two main ideas can be concluded.

Firstly, classical states description is fundamentally different from the quantum one. In the classical world, the state of a system can be described using exact values of position and momentum. Quantum physics on the other hand, describes a state using a wave function, which can represent probabilities for the measurement outcomes of observables such as position and momentum.

Secondly, in the classical field, the behavior of each particle and its interactions with other particles is predictable. What is more, if it is measured twice, the result of the experiment – if the particle is not modified – is unchanging throughout time. Nevertheless, quantum physics is nonintuitive. The relationship between states and measurements is uncertain, and it can change over time. If a particle is measured twice, the solutions obtained can be random and unexpected.
Therefore, quantum mechanics is nondeterministic, meaning that it does not describe the behavior of physical systems with complete precision (is probabilistic).

INTRODUCTION TO QUANTUM MECHANICS

- State vectors, the inner product and the superposition principle

In quantum mechanics, in order to describe the properties of a quantum system at a given time, the state of a system, state vectors are used. They are represented by a complex-valued vector in a vector space called Hilbert space.

To start with, let´s explain how to represent state vectors. We use “bras” and “kets”. They are mathematical objects that are written using angle brackets and bars.
A ket is a column vector which can be represented with a vertical bar on the left and angled bracket on the right, between the name of the vector. In symbols,

$$|\mathrm{A}\rangle = \begin{pmatrix} \mathrm{A}_1 \\ \mathrm{A}_2 \\ \mathrm{A}_3 \end{pmatrix} = \mathrm{A}_1|\mathrm{i}\rangle + \mathrm{A}_2|\mathrm{j}\rangle + \mathrm{A}_3|\mathrm{k}\rangle$$

Where $|\mathrm{A}\rangle$ is the ket representation of the vector $\vec{\mathrm{A}}$ and $\mathrm{A_n}$ are the ket components for the basis ket $|\mathrm{i}\rangle$, $|\mathrm{j}\rangle$ and $|\mathrm{k}\rangle$.

On the flip side, bras are represented by a row vector and are written in opposite positions as kets, with the angled bracket on the left and the bar on the right. They are the conjugate transpose of a ket.

$$\langle A| = (A_1^* \;\; A_2^* \; A_3^*)$$

For every bra there is a corresponding ket. To clarify, bras and kets don´t inhabit the same vector space. Bras inhabit the "dual space" to the space of kets. For each ket in the Hilbert space, exists a corresponding bra in the dual space. This can be hard to understand, so think of the complex conjugates in classical mathematics. There is a corresponding complex conjugate $(1 - 2i)$ for each complex number $(1 + 2i)$. This relationship is similar to the relationship between bras and kets.

When a bra acts on its corresponding ket, the result is the square of the ket's norm, a scalar value.

$$\langle A|A\rangle = (A_1^* \;\; A_2^* \; A_3^*) \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = |\vec{A}|^2$$

This operation is called "the inner product". Mathematically, the inner product of two state vectors gives a complex result, which is defined as the sum of the product of the complex conjugates of the coefficients representing the first vector with the coefficients representing the other vector. Basically, the product of a bra and a ket. It represents the projection of one vector onto the other. In the case below, the inner product can be thought as the projection of the content of the ket $|B\rangle$ onto the content of the bra $\langle A|$.

$$\langle A|B\rangle = (A_1^* \;\; A_2^* \; A_3^*) \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \; A_1^* B_1 + A_2^* B_2 + A_3^* B_3$$

A distinct approach could be explained based on the superposition principle. The superposition principle is a fundamental aspect that states the idea that two or more states can combine and form a new state. To put it another way, suppose there are two pure states, states that can be represented by a single state vector with total certainty, $|\psi_1\rangle$ and $|\psi_2\rangle$, they can be superposed to create a new one, $|\psi\rangle$:

$$|\psi\rangle = c_1 \, |\psi_1\rangle + c_2 \, |\psi_2\rangle$$

The final equation is a weighted combination, where c_1 and c_2 are the components of $|\psi\rangle$, which is a mixed state.

So, the inner product can be understood as representing how much of state $|\psi_1\rangle$ is present in $|\psi\rangle$. Mainly, it is used to measure the similarity between the two states, so it determines the relative weight of the states in the superposition.

This idea allows us to define the components of $|\psi\rangle$, c_1 and c_2, as the inner product between the combined state and the corresponding state vector $|\psi_1\rangle$ or $|\psi_2\rangle$ under one condition. The state vectors must be orthogonal.

In symbols,

$$\langle\psi_1|\psi_2\rangle = 0$$

It can be proved easily:

$$|\psi\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle$$

We take the inner product of both sides with $\langle\psi_1|$:

$$\langle\psi_1|\psi\rangle = c_1\,(\langle\psi_1|\psi_1\rangle) + c_2\,(\langle\psi_1|\psi_2\rangle)$$

If ψ_1 is normalized, $\langle\psi_1|\psi\rangle = 1$. Furthermore, if ψ_1 and ψ_2 are orthogonal:

$$\langle\psi_1|\psi\rangle = c_1$$

The components are complex coefficients that do not have a direct physical meaning by themselves, but are needed in order to calculate the probability of finding the system in a particular state. The larger the magnitude of the component is, the higher the probability is going to be.

In this case, the square root of the magnitude of c_1 will describe the probability of finding the system in the $|\psi_1\rangle$ state:

$$P_{\psi 1} = |c_1|^2$$

There are several different ways to assess the probability. For instance, it is represented as:

$$P_{\psi 1} = c_1{}^* c_1$$

Likewise, this applies to the c_2 coefficient. The probability of being in the $|\psi_2\rangle$ state can be written as $c_2{}^* c_2$.

As the inner product can define the components, it can also be used to calculate probabilities.

$$P_{\psi 1} = |c_1|^2 = |\langle\psi_1|\psi\rangle|^2$$

- Expectation values

The expectation values describe the average over the possible outcomes of a measurement of an observable in a system. They are usually represented as a symbol called “E” or “$\langle\ \rangle$”.

In terms of mathematics, the expectation value of an observable A is defined by the inner product:

$$\langle \mathrm{A} \rangle = \langle \psi | \hat{\mathrm{A}} | \psi \rangle$$

Represented by the operator $\hat{\mathrm{A}}$ for a system in the state $|\psi\rangle$.

An operator is a mathematical object that acts on a state vector of a system and produces another state vector. From a mathematical perspective,

$$\hat{\mathrm{A}}|\psi\rangle = |\phi\rangle$$

$|\phi\rangle$ also belongs to the Hilbert space, as $|\psi\rangle$ does it.

The result of the expectation value will be the average value of all measurement outcomes of the observable A when operating $\hat{\mathrm{A}}$.

One important idea to take into consideration is that the expectation value is not the result of one measurement, but the average over many evaluations. In other words, the expected value is a measure of the most likely value of a quantum mechanical observable. It is not the result of an experiment over time, nor is it the specific numerical value obtained in a single measurement. Instead, it represents the average of an observable's value over many measurements, weighted according to the probability of each value occurring.

Another way to calculate the expectation value is by using the probabilities of the overcomes, and the overcomes themselves.

$$\langle \mathrm{A} \rangle = \sum_{\mathrm{n}} \lambda_{\mathrm{n}}\, \mathrm{P}(\lambda_{\mathrm{n}})$$

$\mathrm{P}(\lambda_{\mathrm{n}})$ is the probability of the eigenvalues λ_{n} of the operator being the outcome of the measurement.

To comprehend this, let´s explain a fundamental property of quantum mechanics. The eigenvalues of an operator represent the possible outcomes of a measurement of the corresponding observable. For this reason, the expectation value is the weighted sum of the possible results, where each outcome is weighted by its corresponding probability.

- Commutation

Considering two observables, A and B, the order in which the measurements are made doesn´t affect the final results, if and only if, both observables commute. The degree of commutation is defined as:

$$[\mathrm{A},\mathrm{B}] = \mathrm{AB} - \mathrm{BA}$$

The more sensitive the observables are to the order in which the experiment is made, the larger the commutator. If the order does not matter, the commutator will be zero.

It's worth mentioning that $[\mathrm{A},\mathrm{B}] = -[\mathrm{B},\mathrm{A}]$.

Another important concept related to commutation is overlap. It indicates how much the outcomes of two observables after measuring coincide with each other. In simpler terms, if the overlap is perfect, the results obtained are completely compatible. Measuring one observable does not affect the subsequently measurement of the other one. Therefore, they commute.

On the other hand, if the overlap is not perfect, the measurement outcomes are mutually exclusive. It affects to the probability of analyzing one observable after the other, which means that they do not commute. Consequently, non-commuting operators can result in uncertainty or unpredictability in the results.

When two observables belong to two different systems, they can construct a new combined system by taking the tensor product of the states of the individual systems. This mathematical operation is notated as:

$$|\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle$$

Where $|\psi\rangle$ and $|\phi\rangle$ are states of the systems, respectively.

Despite the confusing notation, it is important to remark that $|\psi\phi\rangle$ represents a single state of the composite system, not two, even though it is written with a double index.

The main feature of a product state is that each subsystem behaves independently of the other. This means that the state of each subsystem does not depend on the state of the other subsystem, and measurements on one subsystem do not affect the state of the other subsystem.

Nevertheless, if the two systems are correlated, the state of one system can influence the state of the other systems, and vice versa.

For this reason, when we try to combine the two correlated systems, an entangled system is obtained.

The latter is defined as a system in which the state of one of the systems depends on the state of the other one, even if the two systems are spatially separated. If the state of one particle is measured, the state of the other particle is instantaneously influenced. This idea is a consequence of the superposition principle, which states that a quantum system can exist in multiple states simultaneously.

In order to understand the entangled systems, we can think in classical world. Imagine flipping two coins, 1 euro and 2 euros, simultaneously. If one coin, the 1 euro or 2 coin (doesn´t matter), comes up heads, the second coin will either come up heads or tails. The only way we can tell which, is by looking at it. To put it another way, the state of one coin has no effect on the state of the other. It doesn´t affect the second coin´s state. Therefore, the coins are not entangled.

However, consider two quantum coins. In the same way, one of one euro and the other of two. Let´s think about you and me. Imagine that a friend gives one of the coins to each of us, but we cannot look at them yet. First, you need to go to your dream destination. Free vacation. Once you are there and I am here, at home (not fair), I am going to call you and tell you that if I guess which one is yours, the one euro coin or the other, you have to invite me to your hotel so we can both enjoy the holidays. Obviously, I will win. Why? Because in the moment that I look at my coin, I know yours perfectly. It would be impossible to guess before looking at the coin, but after it is 100% certain.

Even if the coins are located far apart, we can instantaneously determine the state of one coin by measuring the other. The two coins' states are connected. Here, the condition of one coin immediately influences the condition of the other coin. Therefore, the coins are entangled.

This “stupid” example shows how entanglement works. I hope you have understood it.

Returning to the quantum world and leaving the classical world behind, there is an important idea we must know. Entanglement is not all or nothing.

The states of entangled particles can be more or less entangled. The degree of entanglement is determined by the amount of information that can be transferred from one particle to another through the entangled state. If they are highly entangled, the information we know about the other state is very significant. The stronger the correlation between the particles, the more information can be transferred and the more entangled the particles are considered to be.

An example of a maximally entangled state is called the singlet. It can be express as:

$$|\text{sing}\rangle = \frac{1}{\sqrt{2}} \left(|\text{ud}\rangle - |\text{du}\rangle\right)$$

Where $|\text{ud}\rangle$ and $|\text{du}\rangle$ are states of a combined system.

The singlet state is a state in which two particles are in a superposition of being spin-up and spin-down, as seen.

On the other hand, the triplet states are another example of highly entangled states. They are described mathematically:

$$|T_1\rangle = \frac{1}{\sqrt{2}} (|ud\rangle + |du\rangle)$$

$$|T_2\rangle = \frac{1}{\sqrt{2}} (|uu\rangle + |dd\rangle)$$

$$|T_3\rangle = \frac{1}{\sqrt{2}} (|uu\rangle - |dd\rangle)$$

To close with, two fundamental statements need to be mentioned.

First of all, an entangled state is a full description of two or more particles that are connected and their behavior affects each other. No further information can be known about the individual particles.

Secondly, in a maximally entangled state, nothing is known about the individual subsystems.

- Hermitian operators

A Hermitian operator is an operator that is equal to its own adjoint. To clarify, it is equal to the transpose of its complex conjugate.

This can be expressed as:

$$M = [M^T]^*$$

The right side of the equation is also denoted as $M^\dagger$.

This type of operators has several useful properties that will be helpful in further sections.

To begin with, let´s talk about one of the most important properties. The eigenvalues of a Hermitian operator are real numbers.

$$M|\psi\rangle = \psi|\psi\rangle$$

Where ψ is real. In other words, equal to its complex conjugate ψ^*.

It can be easily demonstrated:

Firstly, we need to rewrite the last equation so we have the bra $\langle\psi|$ applied to the Hermitian conjugate of M. Using the definition of Hermitian conjugation:

$$\langle\psi|\mathrm{M}^{\dagger} = \langle\psi|\psi^{*}$$

Secondly, the inner product must be calculated in both equations:

$$\langle\psi|\mathrm{M}|\psi\rangle = \psi\,\langle\psi|\psi\rangle$$

$$\langle\psi|\mathrm{M}^{\dagger}|\psi\rangle = \psi^{*}\langle\psi|\psi\rangle$$

Thirdly and last, as we know that M is Hermitian,

$$\langle\psi|\mathrm{M}|\psi\rangle = \psi\,\langle\psi|\psi\rangle$$

$$\langle\psi|\mathrm{M}|\psi\rangle = \psi^{*}\langle\psi|\psi\rangle$$

Therefore,

$$\psi\,\langle\psi|\psi\rangle = \psi^{*}\langle\psi|\psi\rangle$$

ψ is equal to its complex conjugate ψ^{*}.

Another important property is that the eigenvectors of a Hermitian operator, corresponding to different eigenvalues, are orthogonal.

Remember that orthogonal means that the inner product of the eigenvectors is equal to zero.

We start with

$$\mathrm{M}|\psi_1\rangle = \psi_1|\psi_1\rangle$$

$$\langle\psi_2|\mathrm{M} = \langle\psi_2|\psi_2$$

Notice that the second equation is valid only because M is Hermitian, and then,

$$\psi_2 = \psi_2{}^{*}$$

If not, we would need to write it as:

$$\langle\psi_2|\mathrm{M} = \langle\psi_2|\psi_2{}^{*}$$

As before, we take the inner product on both equations with the other eigenstate:

$$\langle\psi_2|\mathrm{M}|\psi_1\rangle = \psi_1\,\langle\psi_2|\psi_1\rangle$$

$$\langle\psi_2|\mathrm{M}|\psi_1\rangle = \psi_2\,\langle\psi_2|\psi_1\rangle$$

Substracting the first one from the second one:

$$\psi_1 \langle\psi_2|\psi_1\rangle - \psi_2 \langle\psi_2|\psi_1\rangle = 0$$

If ψ_1 and ψ_2 are different,

$$\langle\psi_2|\psi_1\rangle = 0$$

Orthogonality has been demonstrated.

This characteristic is crucial because it enables the linear combination of orthogonal eigenvectors representation of a quantum state, which in many occasions, can simplify calculations.

- Unitary and time-development operator

The evolution operator $U(t)$ is used in quantum mechanics to describe how a quantum system evolves over time. Given the initial state of a system, the time-evolution operator can be used to calculate the state of the system at any later time.
Imagine there is a state $|\psi\rangle$. In order to distinguish it in different moments, let´s denote $|\psi(0)\rangle$ for the state $|\psi\rangle$ at time 0, initial time. On the other hand, at time t, it will be described as $|\psi(t)\rangle$.

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

It is important to note that the state does not represent the actual state of the system, but rather, it gives us the probability amplitude of finding the system in a particular state. Only when a measurement is made we can determine the actual state of the system. The outcome of the measurement is uncertain, and the probabilities of obtaining different results are given by the square of the magnitude of the quantum state. So even if we know the quantum state at a specific time, we can't say for sure what the state of the system is.

The state only gives us information about the probabilities of finding the system in different states, and only a measurement can determine the actual state.

The time-development operator has several important properties:

Time reversibility: Because the operator is time-reversible, we may compute the state of the system at any other time if we know the state of the system at a given time. This enables both future and previous predictions.

This property is quite obvious, knowing all that we already know.

Linearity: The evolution operator is linear. It can be represented as a linear combination of other evolution operators.

If we have two time-development operators $U_1(t)$ and $U_2(t)$, a third one can be represented:

$$U(t) = \alpha U_1(t) + \beta U_2(t)$$

The conservation of distinctions: The inner product between two states will remain constant over time.

Essentially, if the inner product between two states is equal to 1, the states are said to be "distinguishable", which implies that the states are orthogonal. On the other hand, if the inner product is equal to 0, the states are said to be "indistinguishable".
The conservation of distinction means that if two states are distinguishable at one time, they will continue to be distinguishable at all times, and the same logic for indistinguishable states. If the inner product between two states is equal to zero, it will remain zero over time.

It can be represented with symbols:

$$\langle \psi(0) | \psi(0) \rangle = 0$$

Therefore,

$$\langle \psi(t) | \psi(t) \rangle = 0$$

Unitarity: The evolution operator is unitary.

The unitarity of U ensures that the evolution it describes is norm-preserving, meaning that the square of the norm of the wave function representing a state is always equal to the probability of finding the system in that state. This ensures that the total probability of finding the system in any state is always equal to 1.

As the norm of the quantum state remain constant, the total probability of finding the system in any of its possible states is conserved over time.

Unitary can be demonstrated mathematically as:

$$\langle i | U^{\dagger}(t) U(t) | j \rangle = \delta_{ij} \; ; \quad i = j$$

Where $U^{\dagger}(t)$ is the adjoint of $U(t)$, and δ_{ij} is the Kronecker delta function, which is equal to 1 if $i = j$ and 0 otherwise, being i and j basis vectors.

$$\langle i | U^{\dagger}(t) U(t) | j \rangle = 0 \; ; \quad i \neq j$$

Likewise, we know that

$$\langle i | j \rangle = \delta_{ij} \; ; \quad i = j$$

$$\langle i|j\rangle = 0 \; ; \quad i \neq j$$

Furthermore,

$$\langle i|I|j\rangle = \delta_{ij} \; ; \quad i = j$$

$$\langle i|I|j\rangle = 0 \; ; \quad i \neq j$$

Consequently, $U^{\dagger}(t)U(t)$ acts as the identity operator between any two basis vectors in the basis set.

These equations above state that the inner product of two time-evolved states with the same initial state is equal to the identity matrix, implying that the evolution is reversible and that the total probability is conserved.

- The Hamiltonian and time-dependent Schrödinger equation

The evolution operator $U(t)$ studied above describes the time-evolution of the quantum state over a finite interval of time t. However, to reach the Schrödinger equation for the time-evolution of a quantum state, it is often more convenient to break up the time interval t into many infinitesimal time intervals ϵ and consider the time-evolution operator $U(\epsilon)$ for each small interval.

This allows us to build up a finite time interval by combining many infinitesimal intervals. In other words, we can rewrite $U(t)$ as a combination of $U(\epsilon)$.

Let´s focus on our goal, solve the Schrödinger equation.

We define the evolution operator in symbols:

$$U(\epsilon) = I - i\epsilon H$$

The minus sign is arbitrary. We just use a "-" because it will be more helpful, it will provide an easier approach in the future. H is known as the Hamiltonian, which will be explain in a few sections.

Once we have defined $U(\epsilon)$, we calculate the Hermitian conjugation.

$$U(\epsilon)^{\dagger} = I + i\epsilon H^{\dagger}$$

As $U(\epsilon)\, U(\epsilon)^{\dagger} = I$,

$$(I - i\epsilon H)(I + i\epsilon H^{\dagger}) = I$$

If we ignore terms with ϵ^2 or higher powers of ϵ, the equation leads to:

$$H^\dagger - H = 0$$

Hence, the Hamiltonian is Hermitian.

$$H^\dagger = H$$

Returning to the goal, we remember $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, describing the state $|\psi(t)\rangle$ in terms of it at an initial time 0.

We got rid of t so, we can rewrite the expression:

$$|\psi(\epsilon)\rangle = U(\epsilon)|\psi(0)\rangle$$

To put it differently,

$$|\psi(\epsilon)\rangle = (I - i\epsilon H)|\psi(0)\rangle$$

Modifying the order and taking the limit as ϵ approaches 0,

$$\frac{\partial|\psi\rangle}{\partial t} = -iH|\psi\rangle$$

Finally, we reach the time-dependent Schrödinger equation.

- Quantum statistics

Let's move on to another important concept called density operator. It represents the quantum state of a system. In other words, it encodes all the information about the state of a system, including the probabilities of the measurements results. It can describe both pure and mixed states, in contrast to the wave function, which only uses pure states.

The operator ρ is defined as the outer product of the state vector of the system with itself:

$$\rho = \sum_n \rho_n |\psi_n\rangle\langle\psi_n|$$

$|\psi_n\rangle$ being the states of the system and ρ_n their coefficients.

The elements of the density operator represent the probability amplitudes of the different states of the system, and the diagonal elements represent the probabilities of the system being in those states.

The density operator, also known as density matrix, is also useful to calculate the expectation value in terms of the trace.

$$\langle A \rangle = \mathrm{tr}\{\rho \hat{A}\}$$

The trace returns the sum of the diagonal elements of the matrix. In the formula above, the trace is being used to sum the elements of the matrix formed by the product of the density operator and an observable A.

Remember we studied before that the expectation value was $\langle A \rangle = \sum_n \lambda_n P(\lambda_n)$. Consequently,

$$\sum_n \lambda_n P(\lambda_n) = \mathrm{tr}\{\rho \hat{A}\}$$

Another purpose of the trace is to write the probability of a measurement A. It is helpful when we do not have complete knowledge of the state of a quantum system, which is common – due to the uncertainty principle.

To obtain the probability of a state, first, we need to calculate the density operator of the corresponding state. Once this is done, we just need to figure out the trace of that density operator recently estimated, multiplied by the density operator of the system.

$$P(A) = \mathrm{tr}\{\rho_a \rho\}$$

Some fundamental properties of the density matrix are:

1. It is a positive semi-definite matrix. Hence, its eigenvalues are all non-negative.
2. It is Hermitian. This means that it is equal to its own conjugate transpose.

$$\rho = [\rho^T]^*$$

3. It is normalized. The sum of the diagonal elements of the matrix is equal to one.

$$\mathrm{tr}\{\rho\} = 1$$

Any normalized Hermitian operator can be accepted as a valid density operator for a quantum state on condition that it has non-negative eigenvalues.

Moving on to another related subject, let's consider an operator, known as reduced density operator, used to describe the state of a subsystem of a larger system. It is obtained by taking the trace of the density operator of the larger system over the degrees of freedom of the subsystem.

$$\rho_1 = \mathrm{tr}\{\rho\}$$

ρ_1 refers to the reduced density operator of the subsystem 1. $\mathrm{tr}\{\rho\}$ instead, denotes the trace over the degrees of freedom of the composite large system.

On the other hand, one important idea you must keep in mind is that the states of a system can be pure or mixed, as it has been mentioned before. A pure state can be represented by a single state vector, as a result of this, there is no uncertainty in the state. A mixed state, on the other hand, is a state that cannot be represented by a single state vector. Instead, it is a probabilistic mixture of several pure states.

With the aim of characterize the purity of a state, several options can be selected.

Firstly, the von Neumann entropy. It is a measure of the amount of uncertainty or mixedness in a quantum state.

$$S = -k\,tr\{\rho \ln \rho\}\ ;\ k \equiv \text{Boltzmann constant}$$

Pure states are defined as $S = 0$, while for mixed states, the overcome is positive. In terms of the von Neumann entropy: $S > 0$.

Secondly, to measure how mixed a quantum state is, the purity can be used. It is defined as the trace of the square of the density operator, which value is one for pure states, whereas for mixed states, is less than one.

$$\text{Purity} \equiv tr\{\rho^2\}$$

- Principles of Quantum Mechanics

The principles of quantum mechanics are fundamental concepts that describe the behavior of very small objects. We are going to list them, which will also be useful in order to remember key concepts:

Postulate 1: Observables in quantum mechanics are represented by linear and Hermitian operators.

Linearity means that if we have two wave functions, ψ_1 and ψ_2, and an operator $\hat{A}$, then the following must satisfy:

$$\hat{A}\,(c_1\psi_1 + c_2\psi_2) = c_1\,\hat{A}\,\psi_1 + c_2\,\hat{A}\,\psi_2$$

It is required for the superposition property, which holds that a quantum state can be in multiple states at the same time, just before being measured. Once the measurement is made, the entire superposition of states collapses to a single term.

On the other hand, remember Hermitian operators satisfy the following property, where M is an operator:

$$M = M^{\dagger} = [M^{T}]^{*}$$

That is, the operators must be equal to their own Hermitian conjugates.

The overcome of any experiment is a real number. Hence, the observables in quantum mechanics need to be represented by Hermitian operators because they have real eigenvalues, which are the results of the measurement (postulate 2). What is more, Hermitian operators are used because they form a complete and orthogonal set. In other words, any wave function can be expanded as a sum of the eigenvectors of the operator.

Postulate 2: The only possible results of a measurement are the eigenvalues of the operator that represents the observable.

$$\hat{A}|\lambda_n\rangle = \lambda|\lambda_n\rangle$$

This is only certain if the vectors $|\lambda_n\rangle$ associated with the operator $\hat{A}$ don´t change their direction when it is applied to them.

Postulate 3: Unambiguously distinguishable states are represented by orthogonal vectors.

Unambiguously distinguishable states are defined as different states which can be distinguished from each other without uncertainty. That is to say, it is possible to tell the states apart with total accuracy.

Orthogonality refers to the property of two vectors being perpendicular to each other. Two vectors, u and v, are orthogonal if their inner product is zero. In symbols:

$$\langle u|v\rangle = 0$$

Orthogonal vectors are completely independent of each other. As a result, we can determine which of the two states the system is in by measuring it with complete certainty.

Postulate 4: The probability of a quantum system being in a particular state is determined by the squared magnitude of the corresponding coefficient in the state vector. This can be written as an equation, where $|A\rangle$ is the state vector, and $|u\rangle$ and $|d\rangle$ are possible states where the system can be:

$$|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle\,; \quad \begin{cases} P_u = |\alpha_u|^2 \\ P_d = |\alpha_d|^2 \end{cases}$$

It can also be expressed as the product of the inner product of $|u\rangle$ and $|A\rangle$ and its complex conjugate.

$$P_u = \langle u|A\rangle\langle A|u\rangle$$

The inner product of two vectors is a measure of the overlap between the vectors.

To express it differently, it estimates the degree of alignment between the two vectors. If they are perpendicular, the result will be equal to zero. However, if they are perfectly aligned, then the result will be the product of the magnitudes of the vectors.

The expression can be simplified as:

$$P_u = |\langle u|A\rangle|^2$$

The larger the magnitude squared of the inner product $|\langle u|A\rangle|^2$, the greater the probability of finding the system in state $|u\rangle$.

The inner product gives us a measure of the overlap between the two states, and the larger the overlap between $|u\rangle$ and $|A\rangle$, the more likely it is that the system will be found in state $|u\rangle$ upon measurement.

Postulate 5: The evolution of state-vectors with time is unitary.

The state vector of a quantum system evolves over time in accordance with the Schrödinger equation, which describes how the state of a system changes as time passes. It can be written as:

$$i\hbar \frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle$$

$|\Psi\rangle$ is a quantum state and H is a Hamiltonian operator (it will be described latter, as said).

The expression allows to figure out how the state vector will be at any later time if the initial state vector and the Hamiltonian operator are known.

The Schrödinger equation is unitary, which means that it preserves the inner product of state vectors. In other words, if two states are initially orthogonal, they will remain orthogonal with the passage of the time.

2. QUANTUM OPTICS

Quantum optics and quantum mechanics are closely related, but they are not exactly the same.

Quantum mechanics is a fundamental theory that explains the behavior of all quantum systems. It is a set of principles that provides a description of matter and energy at the quantum level. On the other hand, quantum optics is focused specifically on the study of light and how it interacts with matter.

Quantum optics is based on quantum mechanics, as it uses concepts from quantum mechanics, in addition to electromagnetism and other fields.

QUANTUM FIELD THEORY OF LIGHT

Light can be described as a field, an electromagnetic field. This can be explained because the observables extend over time and space. The values of the physical quantity being measured, in this case the electromagnetic field of light, can be measured at every point in space and time. Basically, light can be measured at any point in space and time.

We know that the field is composed of electric and magnetic fields, and that both oscillate perpendicular to each other, and at the same time, they oscillate perpendicular to the direction of propagation of the wave. This allows light to be described as an electromagnetic wave.

We assume that the classical fields are the ensemble averages of the quantum fields:

$$\mathbf{E} = \langle\psi|\hat{\mathrm{E}}|\psi\rangle \; ; \; \mathbf{H} = \langle\psi|\hat{\mathrm{H}}|\psi\rangle$$

The electric field $\mathbf{E}$ is the expectation value of the observable $\hat{\mathrm{E}}$. This also applies to the magnetic field $\mathbf{H}$ and its observable $\hat{\mathrm{H}}$.

As the classical electromagnetic field obeys the Maxwell´s equations, the quantum field strengths ($\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$) also do. It is important to remark that this does not mean that individual quantum fields obey Maxwell's equations, but only the ensemble average of the quantum fields.

The linearity of the Maxwell's equations allows us to express the fields as expectation values of the corresponding quantum observables. This is because the expectation values of the quantum fields are calculated as the inner product of the state vector and the operator, which is a linear combination of the eigenvectors. As a result, the behavior of the quantum fields can be calculated using the Maxwell's equations by replacing the classical fields with the expectation values of the corresponding quantum observables.

In conclusion, the behavior of the classical fields can be predicted using the ensemble average of the quantum fields.

Furthermore, if the expectation value $\langle\psi|\hat{\mathrm{A}}|\psi\rangle$, of a Hermitian operator $\hat{\mathrm{A}}$ is zero for all states, it will also be zero for its eigenstates, and as the eigenvalues are given by the inner product $\langle\psi|\hat{\mathrm{A}}|\psi\rangle$, all eigenvalues will be zero as well.

- Hamiltonian operator

It is challenging to determinate which quantum operator is the most important, as the significance of the operators depends on the context it is used in. However, there is one which undoubtedly is crucial in quantum optics: the Hamiltonian. The moment to understand it came.

We have already mentioned it, but now we are going to focus on its description.

It governs the time evolution of physical quantities. Basically, it acts on the state vector of a quantum system, describing how the state of the system will evolve over time.

It describes the dynamics of the system and the way in which different physical quantities change over time. Given the Hamiltonian, we can predict the probability of finding a system in different states or different outcomes of measurements at different times. It provides the necessary information for calculating the time-evolution of all observable properties of the system.

The Hamiltonian, H, appears in the time-dependent Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial|\Psi\rangle}{\partial \mathrm{t}} = \mathrm{H}|\Psi\rangle$$

Knowing the Hamiltonian, allow as to solve the formula and compute how the state evolves over time, as it has already been said.

But the Hamiltonian plays a dual role. It is useful, on the other hand, to calculate the energy of a quantum mechanical system. The energy levels that can be observed in a quantum system are the same as the eigenvalues of the H operator.

These levels in a quantum system are not able to take on any continuous value within a range. What does it mean? It means that they cannot be continuous values, they can only be certain discrete values. Which make sense because the eigenvalues of H are real values, as it is a Hermitian operator. The eigenvectors of the Hamiltonian operator, also known as energy eigenstates, correspond to those energy levels.

To find the energy eigenvalues, the time-independent Schrödinger equation must be solved. By definition,

$$\mathrm{H}|\mathrm{E}\rangle = \mathrm{E}|\mathrm{E}\rangle$$

where H is the Hamiltonian operator, E is the energy eigenvalue, and $|\mathrm{E}\rangle$ is the energy eigenstate.

Once the energy eigenvalues and eigenvectors are found, state vector $|\psi\rangle$ can be written as a linear combination of the energy eigenstates, as they form an orthonormal basis:

$$|\psi\rangle = \sum_{\mathrm{n}} \mathrm{c}_{\mathrm{n}}|\mathrm{E}\rangle$$

It's worth noting that this is a general case, where the system can be in a superposition of energy eigenstates. This explains why the state vector is written as a linear combination of the energy eigenstates and not as a single energy eigenstate.

What is more, it´s important to clarify that the representation of the state vector in the energy basis will provide information about the energy levels of the system. However, other basis can provide different or additional information.

The probability of finding the system in a particular energy eigenstate is given by the square of the corresponding coefficient, c_n.

$$P_e = |c_n|^2$$

Since the state vector $|\psi\rangle$ represents the quantum state of a system, it changes with time as the system evolves. To account for this, we use the time-dependent Schrödinger equation, as it describes how the state vector changes over time. To understand why $|\psi\rangle$ can change over time, we need to keep in mind that although the energy states $|E\rangle$ don´t evolve over time (stationary states), the coefficients c_n can do so if the system is not in one of these stationary states.

- Vector potencial and Coloumb gauge

A gauge is a mathematical construct used to describe the behavior of physical systems. Think of it as a set of rules used for describing how a system acts.

The Coloumb gauge allows the study of an electromagnetic field. In this gauge, the vector potential takes an important role. It is a vector field that is related to the electric and magnetic fields through the Maxwell's equations, which represents the electromagnetic field.

The vector potential $A(r,t)$ is chosen such that the divergence of the vector potential is set to zero. In symbols,

$$\nabla \cdot A(r,t) = 0$$

The vector potential is transverse. Hence, it only has a non-zero component in the direction that is perpendicular to the direction of propagation, perpendicular to the direction of the electric field. That’s the reason why this gauge is useful because it allows to split up the longitudinal and transversal part of the vector.

Charged particles interact with electromagnetic fields, and can be study through the vector potential $A(r,t)$. Essentially, it can describe the behavior of charged particles in electromagnetic fields. For example, the energy levels of an electron in a magnetic field.

The Coulomb gauge condition ensures that the vector potential describes a well-defined electromagnetic field that is proportional to the charge density of the system, as it is related to the charge density ρ through the Poisson equation:

$$\nabla^2 \cdot A(r,t) = \frac{-4\pi\rho}{c^2}$$

Basically, the change in the vector potential is proportional to the charge density of the system.

The electromagnetic wave equation is given by:

$$\frac{1}{\varepsilon}\nabla\times\frac{1}{\mu}\nabla\times\hat{A} + \frac{1}{c^2}\frac{\partial^2\hat{A}}{\partial t^2} = 0$$

- Light modes

In quantum optics, instead of describing the electromagnetic field in terms of electric and magnetic field, as showed above, the central model used in order to describe light is the light mode.

We have already talked about the vector potential and what it is. Now, we will continue defining it in a different way. The $\hat{A}(r,t)$ vector potential can be expressed in terms of the modes of the electromagnetic field, by writing it as a superposition of different modes $A_k(r,t)$, where K is a label that identifies the mode. The mode expansion of the vector potential is given by:

$$\hat{A}(r,t) = \sum_k (\hat{a}_k A_k(r,t) + \hat{a}_k^\dagger A_k^*(r,t))$$

In the equation, $\hat{a}_k$ are $\hat{a}_k^\dagger$ are the expansion coefficients that illustrate the amplitude of each mode in the superposition. The coefficient of the complex conjugate wave $A_k^*(r,t)$, $\hat{a}_k^\dagger$, is the Hermitian conjugate of the coefficient of $A_k(r,t)$. This is coherent because the potential vector is Hermitian.

One important idea to pay attention to is that it is possible to depict the electromagnetic field in terms of its modes choosing any set of modes, with the condition that they must form a complete set and obey laws of electromagnetism.

For the purpose of measuring the degree to which two modes differ from each other, the scalar product can be used. We already know this idea, but now we are going to use this tool for the comparation of different modes.

$$(A_1, A_2) = \frac{1}{i\hbar}\cdot\int (A_1^* D_2 - A_2 D_1^*) dV \; ; \; D = -\varepsilon_0\varepsilon\frac{\partial \mathbf{A}}{\partial \mathbf{t}}$$

In this specific form of scalar product, A_1 and A_2 are the two modes of the

electromagnetic field, two mode functions, A_1^* is the complex conjugate of the first mode, D_1 and D_2 are the time derivatives of the modes, and the integral is taken over all space.

The scalar product between the two modes holds some different properties. For instance: linearity and conjugate symmetry.

$$(A_1, A_2) = (A_2, A_1)^*$$

$$(A_0, \alpha_1 A_1 + \alpha_2 A_2) = \alpha_1 (A_0, A_1) + \alpha_2 (A_0, A_2)$$

Another feature of the product is that it obeys

$$(A_1^*, A_2^*) = - (A_1, A_2)^*$$

However, it is not positive definite, the scalar product between one mode and itself can be negative or equal to zero.

The scalar product of modes is a conserved quantity, which means that it does not change over time, it remains constant. This can be mathematically shown by taking the derivative of the scalar product with respect to time:

$$\frac{\partial(\mathbf{A_1},\mathbf{A_2})}{\partial \mathbf{t}} = 0$$

This is a result of the time-independence of the wave equations of the modes.

The symmetry of the scalar product between two modes is based on the idea that the wave equations that describe the modes are real and do not depend on the global phase factor of the modes. This means that the mathematical form of the wave equations remains the same, regardless of the phase factor applied to the modes. This symmetry, known as the gauge symmetry, allows for the multiplication of the modes by an arbitrary global phase factor without affecting the physical results, and it is a key property that ensures the conservation of the scalar product and the total probability in quantum mechanics.

It's worth noting that the scalar product also plays a crucial role in determining the commutation rules for the mode operators $\hat{a}_k$ and $\hat{a}_k^\dagger$. These operators are used to create and destroy photons in a specific mode and their behavior is governed by the commutation relations, which will be developed further in the following section.

- Base commutation relations

Firstly, suppose there are two different modes, A_k and $A_{k'}$, that are orthonormal (called normal modes).

$$(A_k, A_{k'}) = \delta_{kk'} \; ; \; (A_k, A_{k'}^*) = 0$$

Secondly, we are going to define the expansion coefficients in terms of the modes.

$$\hat{a}_k = (A_k, \hat{A}) \quad \hat{a}_k^\dagger = -(A_k{}^*, \hat{A})$$

The next step is, using the definition of the scalar product, calculate the commutator between $\hat{a}_k$ and $\hat{a}_k^\dagger$:

$$[\hat{a}_k, \hat{a}_k^\dagger] = \frac{1}{i\hbar} \cdot \int (A_k * D_k' - A_k' D_k^*) dV$$

Finally, the orthonormality conditions lead to the Bose commutation relations.

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}; \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

These operators, $\hat{a}_k - \hat{a}_k^\dagger$, describe the creation and annihilation of photons in a particular mode k.

To understand the quantum physics of a mode, the only thing we need to know is the Bose commutation relation, which governs the behavior of the mode operators. This commutation relation is a key aspect of the quantization of the electromagnetic field, and it determines how photons interact with each other and with matter. By using the Bose commutation relation and the mode function, one can derive the quantum properties of light modes, such as their energy levels and the statistical behavior of photons in those modes.

The degree of freedom is the number of independent parameters that define the configuration of the electromagnetic field in quantum. This number is equal to the number of modes. Each mode represents a specific frequency and spatial distribution of the field, and the properties of the field, such as the wave amplitudes and quantum fluctuations, are determined by the properties of these modes.
The total Hilbert space of the degrees of freedom is the tensor product of the Hilbert spaces of all the modes. For a system with multiple degrees of freedom, such as an electromagnetic field, each degree of freedom can be described by a separate Hilbert space. The total Hilbert space of the system is then given by the tensor product of the individual Hilbert spaces of each degree of freedom.

The relationship between the Bose commutation relation and the degree of freedom in light can be explained as follows: Each mode of light can be described by a set of creation and annihilation operators that obey the Bose commutation relation. The number of modes of light is equal to the number of degrees of freedom of the light. This means that each mode of light contributes one degree of freedom to the total number of degrees of freedom in the system.

Basically, the Bose commutation relation establishes the relationships between the photons in different modes in a quantum mechanical system.

- Interferences

Wave-like superposition refers to the addition of two or more waves with different amplitudes, phases and frequencies. When two waves are superposed, the amplitude of the resulting wave is the sum of the individual amplitudes, and the expectation value of the field's observable is the sum of the individual expectation values. To see it more clearly, let´s see an example:

Imagine two waves with different amplitudes, frequencies and phases, represented by the complex amplitudes $E_1 = A_1 e^{i\phi_1}$ and $E_2 = A_2 e^{i\phi_2}$. These two waves are superposed, and the resulting wave is $E = E_1 + E_2$. Moreover, the amplitude of the resulting wave is the sum of the individual amplitudes: $|E| = |E_1| + |E_2|$, and the expectation value of the field's observable is the sum of the individual expectation values.

The expectation value of an operator $\hat{A}$ in the resulting wave-like superposition state is given by:

$$\langle E|\hat{A}|E\rangle = \langle E_1|\hat{A}|E_1\rangle + \langle E_2|\hat{A}|E_2\rangle$$

The phase of the resulting wave E depends on the phase difference between the two superposed waves, it can be constructive or destructive.

On the contrary, quantum superposition, refers to the interference of possibilities, or probability amplitudes, of a quantum system. In quantum mechanics, a system can exist in multiple states simultaneously, and the probability of finding the system in a particular state is given by the squared magnitude of the probability amplitude. When two or more states are superposed, the probability amplitude is the sum of the individual amplitudes, and the expectation value of the field's observable is the sum of the individual expectation values. However, this behavior is different from wave-like superposition, as it doesn't add amplitudes directly, but probability amplitudes. An example of this could be:

Consider two possible states of a quantum system, represented by the states $|\psi_1\rangle$ and $|\psi_2\rangle$. The superposition state can be written as $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$, where α and β are complex numbers called coefficients. The expectation value of an operator $\hat{A}$ in this state is given by:

$$\langle\psi|\hat{A}|\psi\rangle = \alpha\,\langle\psi_1|\hat{A}|\psi_1\rangle + \beta\,\langle\psi_2|\hat{A}|\psi_2\rangle$$

It is important to note that in general, the expectation value in a quantum superposition does not behave like a wave-like superposition, unless the states $|\psi_1\rangle$ and $|\psi_2\rangle$ belong to different modes. In that case, they are two independent waves and can add up as in wave-like superposition. And the expectation value $\langle\psi|\hat{A}|\psi\rangle$ will be the sum of the amplitudes $\langle\psi_1|\hat{A}|\psi_1\rangle$ and $\langle\psi_2|\hat{A}|\psi_2\rangle$.

Now, we are going to move on to the idea of mode expansion, which is a way of describing the electromagnetic field in terms of its different modes of oscillation.

It states that one mode A_k can be represented as a linear combination of other modes $A'_{k'}$, which means that it can be thought of as a superposition of these different modes. This is written as:

$$A_k(r,t) = \sum_{k'} A'_{k'}(r,t)\ B_{k'k}$$

Where $B_{k'k}$ are complex constants, that determine how much of each mode $A'_{k'}$ is present in the mode A_k.

This can happen, for example, when light is reflected or transmitted through a medium. In this situation, the incoming modes A_k are different from the outgoing modes $A'_{k'}$, and the reflection or transmission process generates a superposition of these different modes. This is because the reflection and transmission process changes the amplitude and phase of the light waves, creating a superposition of the different modes of oscillation.

The superposition we have already defined can be described in terms of new mode operators $\hat{a}'_{k'}$, which are a linear combination of the original mode operators $\hat{a}_k$, with coefficients $B_{k'k}$.

$$\hat{a}'_{k'} = \sum_k B_{k'k}\,\hat{a}_k$$

In summary, this allows us to describe the electromagnetic field in terms of the new modes $A'_{k'}$ and the associated mode operators $\hat{a}'_{k'}$, preserving the orthonormality of the modes.

- Zero point energy and Casimir force

Let´s consider a cavity made of perfectly conducting, infinitely large plates at a certain distance a, where the electromagnetic field is confined. All the modes of the field will be stationary, because the perfectly conducting plates act as perfect mirrors, reflecting all the electromagnetic waves back into the cavity.

In order to calculate the total energy, the Hamiltonian must be used.

$$\hat{H} = \frac{1}{2}\int (\hat{\mathbf{E}}Đ + Ƀ\hat{\mathbf{H}})dV$$

Knowing that

$$-\nabla\cdot(\mathbf{A}\times\mathbf{H}) + \mathbf{B}\cdot\mathbf{H} = \mathbf{A}\frac{\partial\mathbf{D}}{\partial\mathbf{t}}$$

and that the integral of $\nabla\cdot(\mathbf{A}\times\mathbf{H})=0$ vanishes,

$$\hat{H}=\frac{1}{2}\int(\hat{E}\hat{D}+\mathbf{A}\frac{\partial\mathbf{D}}{\partial\mathbf{t}})dV$$

After some development of the formula, we finally obtain:

$$\hat{H}=\sum_k\frac{\hbar\omega_k}{2}\left(\hat{a}_K\hat{a}_K{}^{\dagger}+\hat{a}_K{}^{\dagger}\hat{a}_K\right)$$

From Bose commutation relations,

$$\hat{H}=\sum_k\hbar\omega_k(\hat{a}_K{}^{\dagger}\hat{a}_K+\frac{1}{2})$$

The total energy of the electromagnetic field is the sum of the energies of the stationary modes.

The minimal value of the energy is then,

$$E_0=\sum_k\frac{\hbar\omega_k}{2}=\infty$$

Therefore, the energy of the electromagnetic field in the vacuum state is infinite, even if there are no photons.

It's important to note that this is a simplifying assumption. In reality, the plates are not perfect conductors and not infinitely large. So, the number of modes will be limited by the finite size of the plates and the imperfections of the material. Anyways, we obtained that the energy of the vacuum state is infinite.

You are probably asking: “what is the vacuum state?”. The vacuum state or quantum vacuum is, according to its name, a state of empty space. However, it is not truly empty. Instead, it is filled with fleeting electromagnetic waves and particles that pop into and out of the quantum field, generating energy. This energy is known as the zero-point energy.

It is described as the lowest possible energy that a system can have. However, it doesn´t mean that it is equal to zero, as we have already demonstrated.

So, we have learnt that in quantum mechanics, even in the absence of any real particles, the vacuum state still has a certain amount of energy associated with it. This is known as the zero-point energy. But why does it have energy? The answer I´m sure is going to sound familiar: the Heisenberg uncertainty principle.
It states that the more precisely the energy of a system is known, the less precisely the time at which that energy is measured can be known, and vice versa. In the case of the vacuum state, which is a state with no real particles, the uncertainty in the energy is

not zero, but rather a certain minimum value. This minimum value is the zero point energy.

The total zero-point energy of the modes of a quantized field in a finite-size cavity is indeed infinite, but this infinity can depend on the size of the cavity. The zero-point energy of a mode is inversely proportional to the size of the cavity. This means that as the size of the cavity increases, the zero-point energy of the modes decreases.

Mathematically, it can be shown that the frequency of a mode in a cavity is inversely proportional to the size of the cavity. This is because the wavelength of the mode is determined by the size of the cavity, and the frequency is inversely proportional to the wavelength.

Given the parameters l, m, and n, and the dimensions a, b, and c of the rectangular cavity, the formula for the frequencies of the modes can be written as:

$$\omega_{lmn} = \sqrt{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}$$

Where l, m, and n are positive integers and a, b, and c are the dimensions of the cavity in the x, y, and z directions, respectively.

In conclusion, the formula describes the frequency of an electromagnetic wave in the cavity for a given set of indices l, m, and n. It shows that the frequency of a mode is inversely proportional to the size of the cavity because as the dimensions of the cavity increase, the frequency of the mode decreases. It has been demonstrated that as the size of the cavity increases, the zero-point energy of the mode decreases.

On the other hand, as the number of modes in the cavity increases, the total zero-point energy increases. This is why the total zero-point energy can be infinite even if the energy of each mode is finite.

Moving to another topic, fluctuations are random variations in the energy of a system. These fluctuations can be caused by the zero-point energy of a quantized field.

In the case of the electromagnetic field, these fluctuations are known as vacuum fluctuations. The fluctuations result in random variations in the electric and magnetic fields, even in the absence of any real charges or currents. This means that even in a perfect vacuum, there is still some residual electric and magnetic field present, and these fields fluctuate randomly due to the zero-point energy of the electromagnetic field.

It's worth noticing that the zero-point energy fluctuations in the electromagnetic field affect the fields, not the particles or the charges. Therefore, it is the electric and magnetic fields that fluctuate as a result of the zero-point energy, not the charges or the particles.

The zero-point energy is a fundamental property of the electromagnetic field, and it cannot be eliminated or reduced. Consequently, these fluctuations are a fundamental feature of the electromagnetic field, and they cannot be avoided.

One important idea to take into consideration is that not all quantum systems have the same level of fluctuations. Some systems have a higher zero-point energy and therefore exhibit larger fluctuations, while other systems have a lower zero-point energy and exhibit smaller fluctuations. The degree of fluctuations depends on the system and the specific quantum state it is in.

Also, it's important to notice that the presence of fluctuations doesn't imply that the system is unstable or that it can't be controlled. In fact, fluctuations can be controlled and reduced by applying external forces to the system.

Vacuum fluctuations have important practical implications in several areas of physics, such as in the Casimir effect.

The Casimir effect states that two parallel, uncharged, and perfect conductive plates in a vacuum experience an attractive force due to the vacuum fluctuations of the electromagnetic field.

When two parallel conductive plates are placed in close proximity, the vacuum fluctuations of the electromagnetic field between the plates are affected by the presence of the plates. The plates act as mirrors, reflecting some of the virtual photons back into the space between the plates. This means that there are fewer virtual photons between the plates than outside of them.

For that reason, the energy density between the plates is lower than the energy density outside of them. This difference in energy density creates a pressure imbalance, and it results in an attractive force between the plates. The force is attractive and its magnitude decreases as the distance between the plates increases.

If we have a movable cavity plate, where the position of the plate is the only variable that changes, the Casimir force can be calculated. To be more precise, an approximation of the Casimir force.

The formula for the mechanical force generated by the zero-point energy of a movable cavity plate is given by:

$$\mathrm{F} = -\frac{\partial \mathrm{U}}{\partial \mathrm{a}}$$

Where F is the mechanical force, U is the difference in zero-point energy between the cavity and empty space, and a is the position of the movable cavity plate. This is derived from the general relationship between force and potential energy, which states that the force acting on an object is equal to the negative gradient of the potential energy with respect to position. In our case, the zero-point energy acts as a

potential energy for the mechanical body (the movable cavity plate), and the force is equal to the derivative of the potential energy with respect to position.

To calculate U, we must follow this equation:

$$U = U_{CAVITY} - U_{FREE\ SPACE}$$

U_{CAVITY} is the zero-point energy inside the cavity with distance a between the plates, and $U_{FREE\ SPACE}$ is the zero-point energy in empty space (outside the cavity).

It's important to note that the zero-point energy density in the cavity is different than that in the empty space, and it's affected by the distance and the boundary conditions of the plates. To calculate the zero-point energy density inside the cavity you need to take into account the specific geometry of the plates and the boundary conditions, and integrate over all the frequency modes of the electromagnetic field.

Also, as it depends on the boundary conditions, if these conditions change, the zero-point energy changes as well, and the difference in zero-point energy between the inside and outside of the cavity will be affected.

Before going further, we need to define the boundary conditions. The boundary conditions in the Casimir force describe the physical conditions that are imposed on the electromagnetic field between two parallel metal plates.

They state that the electric field must be perpendicular to the metal plates and that the normal component of the magnetic field must be zero at the metal surfaces. This means that the electromagnetic field is confined between the plates, it cannot penetrate the metal. Furthermore, only certain wavelengths of the electromagnetic field are allowed between the plates. This restriction of the wavelengths between the plates leads to a reduction in the zero-point energy of the electromagnetic field, which results in the creation of the Casimir force.

The change in the zero-point energy between different plate separations gives rise to the Casimir force. To express it differently, the Casimir force is produced from the difference in the zero-point energy of the electromagnetic field between two parallel metal plates.

When the boundary conditions and the specific geometry of the plates are considered, the difference in zero-point energy between the cavity and empty space can be represented as:

$$U = -\frac{\pi^2 L^2 c\hbar}{720a^3}$$

L^2 is the area of the plates, whereas a is the distance between the plates. I imagine you already know the other terms of the equation, but just in case you don´t, c is the speed of light and $\hbar$ is the reduced Planck constant, $\hbar = \frac{h}{2\pi}$.

This formula is based on the assumption that the plates are perfect conductors, which is not the case for real materials, and the deviation from this formula is significant for small distances.

Now, once we calculated the value of U, we can finally obtain F.

$$\frac{F}{L^2} = -\frac{\pi^2 c\hbar}{240a^4}$$

The formula shows that the force per unit area is proportional to the inverse fourth power of the separation between the plates, a^{-4}. This means that as the separation between the plates increases, the force decreases rapidly.

The negative sign denotes an attractive force between the metal plates, which implies that as the space between them gets smaller, the energy (and hence the force) grows. The formula's application of the reduced Planck constant $\hbar$ shows that the Casimir force per unit area is remarkably low.

This formula relates the change in the zero-point energy of the cavity with the position of the movable plate, assuming that the change in position of the cavity plate is small enough that the cavity can be treated as a harmonic oscillator and that the change in the zero-point energy is linear with respect to position.

To sum up, light is represented as a superposition of multiple electromagnetic modes, each of which is described by the mode functions $A_k(r,t)$. These mode functions have the classical, wave-like properties of light and are subject to the Coulomb gauge, the wave equation, and the necessary boundary conditions.

As we have already mentioned, the Coloumb gauge allows us to describe the electromagnetic field. It specifies the form of the vector potential such that the longitudinal component of the electromagnetic field is zero, and the transverse component is non-zero. What is more, it can be fully described by the vector potential.

On the other hand, the wave equation represents the behavior of light as a wave and determines the propagation of the electromagnetic field in space and time. It contains information about the probability amplitude of finding a particle at a certain location in a certain location at a certain time.

Finally, to conclude the recap, the boundary conditions. The boundary conditions in the Casimir force define the conditions that the electromagnetic field must satisfy at the metal plates, in order to produce the Casimir force. These conditions set limits on the behavior of the electromagnetic field and are used to determine the solutions to the wave equation. For instance, that the electric and magnetic field components must be perpendicular to the metal plates and must be zero at the metal surfaces.

It is important to note that fields are treated as particles and therefore, the electromagnetic field also has quantum degrees of freedom. The electromagnetic field can exist in different states, each with its own unique wave function and probability.

The strength of a particular region of the light field is proportional to the magnitude of the wave function that describes it. In other words, at all points in space and time, the field strength will fluctuate due to changes in the wave function, even when there is no change in the state of the light.

QUANTUM STATES OF LIGHT

The photon number operator ($\boldsymbol{n}$) is a mathematical operator in quantum physics which is used to express the number of photons in a certain mode of the electromagnetic field. In quantized electromagnetic field theory, it acts on the mode amplitudes ($\hat{a}$) to calculate the number of photons in a particular mode.

The creation and annihilation operators, $\hat{a}$ and $\hat{a}^{\dagger}$, respectively, are used to define the photon number operator as $\boldsymbol{n} = \hat{a}^{\dagger}\hat{a}$. The creation operator adds one photon to the number of photons in a mode, whereas the annihilation operator subtracts one photon from the number of photons in a mode.

The expected number of photons in a mode can be computed by using the photon number operator's expectation value regarding a quantum state, represented as $\langle n \rangle$.

$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$ gives the variance of the number of photons in a mode.
This quantifies the spread of a mode's distribution of photons around its average value.

The variance of the number of photons is related to the level of fluctuations in the electromagnetic field, which play an important role in different quantum optical phenomena (any observable events that result from the interaction of light and matter) such as quantum entanglement.

In quantum optics, the photon number operator is used to study the properties of the electromagnetic field in various quantum states, such as coherent states, squeezed states, and Fock states. Don´t worry if you don´t know yet what are these states, we will go through them later in this chapter.

Another important operator is the phase shift operator. It is an operator used in quantum physics to represent a wave function's change in phase.

Mathematically,

$$\hat{U}(\theta) \equiv \exp(-i\theta\boldsymbol{n});\ \theta \equiv \text{phase shift}$$

It is a unitary operator that can act on a wave function or state and adjusts its phase without changing its magnitude.

When acting on $\hat{a}$, results in the amplitude $\hat{a}$ with a phase shift θ.

$$\hat{U}^{\dagger}(\theta)\,\hat{a}\,\hat{U}\,(\theta) = \hat{a}\,\exp(-i\theta).$$

The phase shift operation is a unitary transformation, which means it preserves the normalization of the wavefunction, as we have already mentioned. What is more, as the magnitude of the wavefunction is unchanged, the probability of measuring any particular outcome remains the same after the transformation.

However, the relative phase between different components of the wavefunction can change.

The relative phase between components of the wavefunction determines the pattern of interference between those components, so if it changes, it can cause a different pattern of interference between the components.

The phase factor can be used to describe the effect of time evolution, as well as various other interactions, on the quantum state. In quantum mechanics, time evolution is described by the Schrödinger equation, which states that the time evolution of a quantum state is given by the action of a unitary operator, such as the phase shift operator, on the initial state.

In a nutshell, the phase shift operator is used to characterize the evolution of quantum states over time and to comprehend the influence of phase shifts on the wavefunction.

To finish with, we will talk about two operators, named as quadrature operators.
For monochromatic modes (light fields that have a single frequency), quadrature operators are observables that describe the two components of a quantum light field: the in-phase component and the out-of-phase component.

The in-phase component, represented by $\boldsymbol{q}$, describes the amplitude of the light field, while the out-of-phase component, represented by $\boldsymbol{p}$, describes the phase of the light field.

The quadrature operators can be represented in the form of creation and annihilation operators.

$$\boldsymbol{q} = \frac{1}{\sqrt{2}}\,(\hat{a}^{\dagger} + \hat{a})\;; \quad \boldsymbol{p} = \frac{i}{\sqrt{2}}\,(\hat{a}^{\dagger} + \hat{a})$$

Hence,

$$\hat{a} = \frac{1}{\sqrt{2}}\,(\boldsymbol{q} + i\boldsymbol{p})$$

The creation and annihilation operators are denoted by $\hat{a}^{\dagger}$ and $\hat{a}$, respectively, and they satisfy the commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$.

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{1}{2}[\boldsymbol{q} + i\boldsymbol{p}, \boldsymbol{q} - i\boldsymbol{p}] = \frac{-i}{2}([\boldsymbol{q}, \boldsymbol{p}] + [\boldsymbol{q}, \boldsymbol{p}]) = 1$$

It is not hard to see, then, that $\boldsymbol{q}$ and $\boldsymbol{p}$, which can be thought of as the position and momentum observables for a quantum harmonic oscillator, are canonically conjugate variables:

$$[\boldsymbol{q}, \boldsymbol{p}] = i$$

In this case, the reduced Planck constant, ħ, is set to 1.

This equation states that the position and momentum operators do not commute and are therefore complementary observables. Basically, measurement of one will affect the measurement of the other. This is a result of the Heisenberg uncertainty principle.

On one hand, $\boldsymbol{q}$ and $\boldsymbol{p}$ can be rotated by applying the phase shift operator to them. As we have studied:

$$\hat{q}_{\theta} \equiv \hat{U}^{\dagger}(\theta)\boldsymbol{q}\hat{U}(\theta) = \boldsymbol{q}\cos(\theta) + \boldsymbol{p}\sin(\theta)$$

$$\hat{p}_{\theta} \equiv \hat{U}^{\dagger}(\theta)\boldsymbol{p}\hat{U}(\theta) = -\boldsymbol{q}\sin(\theta) + \boldsymbol{p}\cos(\theta)$$

The relationship between the position and momentum representations of a quantum system can be seen by applying a phase shift of $\frac{\pi}{2}$ to the position or momentum operator. Applying a phase shift of $\frac{\pi}{2}$ to the position operator would bring about the momentum operator.

On the other hand, the quadrature operators can be used in order to describe the energy of an electromagnetic oscillator.

The energy can be described by the equation:

$$\hat{E} = \boldsymbol{n} + \frac{1}{2}$$

Where $\hat{E}$ represents the energy of the oscillator and $\boldsymbol{n}$ represents the number of photons in the mode.

The additional $\frac{1}{2}$ in the equation above represents the zero-point energy of the single mode. The zero-point energy is the minimum possible energy that the oscillator can have, even in its ground state. This energy arises from the uncertainty principle, which prevents the position and momentum of the oscillator from being simultaneously precisely known.

As we know the equivalence between $\boldsymbol{n}$ and $\hat{a}$ ($\boldsymbol{n} = \hat{a}^{\dagger}\hat{a}$),

$$\boldsymbol{q} = \frac{1}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{a}) \quad ; \quad \boldsymbol{p} = \frac{i}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{a})$$

The energy of the electromagnetic oscillator can be expressed as:

$$\hat{E} = \boldsymbol{n} + \frac{1}{2} = \frac{(\boldsymbol{q})^2 + (\boldsymbol{p})^2}{2}$$

STATES OF THE ELECTROMAGNETIC OSCILLATOR

In this section we will introduce four different states of the electromagnetic oscillator, although there are many more. Fock states, quadrature states, thermal states and coherent states.

- Fock states

Fock states are characterized as the eigenstates of the photon number operator.

$$\boldsymbol{n}\,|n\rangle = n|n\rangle$$

The photon number operator acting on a Fock state $|n\rangle$ gives the eigenvalue n, which correspond to the number of photons in the mode.

One important property of Fock states is that they must be orthonormal. In order to ensure the orthonormality of the basis, it is required that the inner product between any two different Fock states is zero and the inner product between a Fock state and itself is one.

This property allow us to calculate probabilities and expectation values in a consistent manner.

We have studied in the previous section that the photon number operator is defined as:

$$\boldsymbol{n} = \hat{a}^{\dagger}\hat{a}$$

Also, we know that $\hat{a}^{\dagger}$ and $\hat{a}$ are called as the creation operator and the annihilation operator, respectively. However, we have never known the reason. Let´s try to demonstrate it mathematically:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

When $\hat{a}$ acts on the photon number operator, it decreases the number of photons in the mode by one, $|n-1\rangle$. Nevertheless, the creation operator $\hat{a}^\dagger$ increases the number of photons in the mode by one, $|n+1\rangle$.

Because of these reasons, the creation operator ("adds" one photon) and the annihilation operator ("eliminates" on photon) are called this way.

You could think now: "What happens if the annihilation operator acts on a state with zero photons?". Good question. We know that if the photon number is an integer, the Fock state $|n\rangle$ represents a mode with n photons. But if the photon number is reduced by one, after each application of the annihilation operator $\hat{a}$, the Fock state will eventually reach $|0\rangle$. To put it simply, it will represent a mode with no photons.

The Fock state $|0\rangle$ is called the vacuum state and is a special state in quantum optics. It is the state with the lowest possible energy and is often used as a reference point for the other Fock states.

The vacuum state is defined by the property:

$$\hat{a}^\dagger\hat{a}|0\rangle = 0$$

The equation above, can be true only if

$$\hat{a}|0\rangle = 0$$

or

$$\hat{a}^\dagger(\hat{a}|0\rangle) = 0 \ ; \ \hat{a}|0\rangle \neq 0$$

From the last equation, after some calculations, we conclude that this possibility must be rejected, because the solution of the equation is not normalizable.

$$|\psi_{-1}(q)\rangle = \hat{a}|\psi_0(q)\rangle$$

It is not hard to figure out that the wave function of the state $|-1\rangle$ is equal to the one of the state $|0\rangle$ when the annihilation operator is applied.

$$\hat{a}^\dagger|\psi_{-1}(q)\rangle = 0$$

Following the same logic, the wave function of the state $|0\rangle$ is equal to the one of the state $|-1\rangle$ when the creation operator acts.

$$|\psi_{-1}(q)\rangle = C\exp\left(\frac{q^2}{2}\right)$$

Finally, in terms of q $|\psi_{-1}(q)\rangle$ is defined as above. Indeed, the result is not normalizable.

Moving forward, analyzing the second equation, we realize that the annihilation operator does not have any effect on the vacuum state and the mode remains in the vacuum state.

We can rewrite the mathematics as

$$\hat{a}|\psi_0(q)\rangle = 0$$

$|\psi_0(q)\rangle$ represents the wave function of the state $|0\rangle$. The result expected should be

$$\hat{a}|\psi_0(q)\rangle = |\psi_{-1}(q)\rangle$$

However, in reality, the state is still the vacuum state.

Considering that the annihilation can be represented in the position representation using the quadrature decomposition and Schrödinger's formula,

$$\hat{a} = \frac{\boldsymbol{q}+i\boldsymbol{p}}{\sqrt{2}}\,;\, \boldsymbol{p} = -i\frac{\partial}{\partial \boldsymbol{q}}$$

The equation can be written as

$$\frac{\boldsymbol{q}+i(-i\frac{\partial}{\partial \boldsymbol{q}})}{\sqrt{2}}|\psi_0(q)\rangle = 0$$

Hence, the wave function of the state $|0\rangle$ is defined:

$$|\psi_0(q)\rangle = \pi^{-1/4}\exp\left(-\frac{q^2}{2}\right)$$

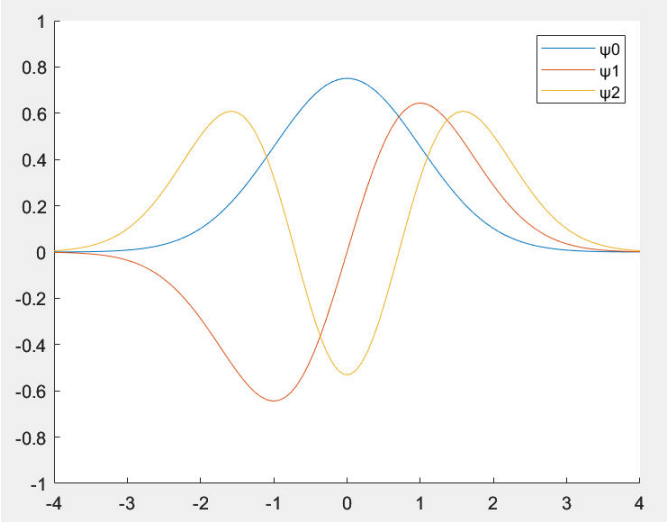
The wave function of the different states can also be described mathematically. In order to achieve it, lets define $|n\rangle$:

$$|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}}|0\rangle$$

After some operations, we can finally express $|\psi_n(q)\rangle$:

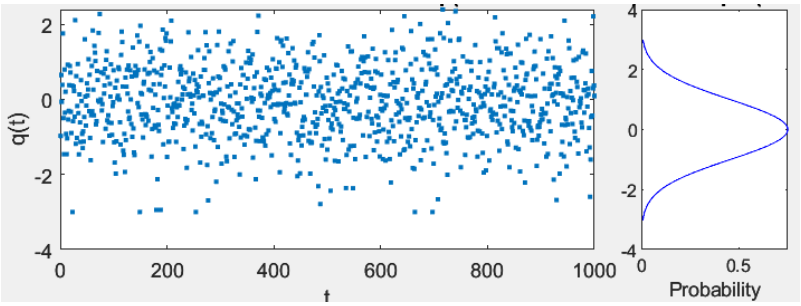
$$|\psi_n(q)\rangle = \frac{H_n(q)}{\sqrt{2^n n!\sqrt{\pi}}}\exp\left(-\frac{q^2}{2}\right)$$

Here the $H_n(q)$ denote the Hermite polynomials.



The quadrature wavefunctions of the first three Fock states: $|\psi_n(q)|^2$

The graphic shows that the quadrature probability distribution for the vacuum state is a Gaussian curve.



The quadrature q of light in the vacuum state

Finally, we can conclude that even if a light mode is considered to be "empty", in the sense that it contains no photons, it can still have a physically meaningful state that can cause physical effects.

This can be explained because the vacuum state of a light mode, represented by the Fock state $|0\rangle$, is not a state of complete absence of light, but rather a state of the lowest possible energy that is still subject to fluctuations inherent to the quantum nature of light.

According to Heisenberg's uncertainty principle, the more precisely we know the position of a particle, the less precisely we can know its momentum, and vice versa. In the case of the electromagnetic field in the vacuum state, this means that the position and momentum quadratures must fluctuate in order to obey the uncertainty principle.

These fluctuations are known as vacuum fluctuations, and the energy they provoke is called the vacuum energy. The latter, is the reason of the $\frac{1}{2}$ term in the energy formula of an electromagnetic oscillator ($\hat{E} = \boldsymbol{n} + \frac{1}{2}$).

The vacuum energy leads to the Casimir force, as we have studied in 2.1.

- Quadrature states

Quadrature states can be thought as the eigenstates of the quadrature operators.

$$\boldsymbol{q}\,|\mathrm{q}\rangle = \mathrm{q}|\mathrm{q}\rangle \;\; ; \;\; \boldsymbol{p}\,|\mathrm{p}\rangle = \mathrm{p}|\mathrm{p}\rangle$$

The notation $|\mathrm{q}\rangle$ and $|\mathrm{p}\rangle$ represents a quadrature eigenstate, where $\boldsymbol{q}$ and $\boldsymbol{p}$ represents the quadrature operators.

These states correspond to the real and imaginary components of the electric field in a light wave. They provide a convenient way to represent the light field in terms of its amplitude and phase. To clarify, the amplitude quadrature state corresponds to the in-phase component of the light field, and the phase quadrature state corresponds to the out-of-phase component of the light field for monochromatic modes.

The quadrature states are considered to be orthogonal and complete.

Orthogonality means that the states are mutually perpendicular, and their inner product is equal to zero. On the other hand, completeness means that any state of the light field can be expressed as a linear combination of these eigenstates.

In symbols,

$$\int_{-\infty}^{+\infty} |\mathrm{q}\rangle\langle\mathrm{q}| = 1 \quad ; \quad \int_{-\infty}^{+\infty} |\mathrm{p}\rangle\langle\mathrm{p}| = 1$$

$$\langle \mathrm{q} \mid \mathrm{q}' \rangle = \delta(\mathrm{q} - \mathrm{q}'), \langle \mathrm{p} \mid \mathrm{p}' \rangle = \delta(\mathrm{p} - \mathrm{p}')$$

The amplitude and phase quadratures of the light field obey the canonical commutation relation. This commutation relation implies that the amplitude and phase quadratures cannot be precisely simultaneously measured, and thus their spectrum must be unbounded and continuous.

This unboundedness of the quadrature spectrum means that the norms of the corresponding amplitude and phase states are not finite.

Hence, since the quadrature states are not truly normalizable, they cannot be used to form a complete orthonormal basis for the light field. Instead, the physical states of the light field are described by coherent states, which are orthogonal and form a complete orthonormal basis for the light field. We will shortly delve into the details of these states, just hold on for a little while longer.

The idea that they are not truly normalizable cause that they cannot be physically realized in an experiment, they don´t have physical meaning. This is because the quadrature states are defined over an infinite range, which would require an infinite amount of energy to generate. As a result, the quadrature states cannot be considered

physical states and are typically used for mathematical convenience and to make predictions about the properties of the light field.

The quadrature wave functions, unlike the quadrature states, have a physical meaning as they represent the amplitudes of the in-phase and out-of-phase components of the light field.

These wave functions are expressed as:

$$\psi(q) = \langle q|\psi\rangle$$

Same idea will apply for $\boldsymbol{p}$. The moduli squared of the quadrature wave functions correspond to the probability distribution of the light field in the amplitude and phase quadratures. These moduli squared provide information about the fluctuations and distribution of the light field in the in-phase and out-of-phase components.

The quadrature eigenstates also follow the Heisenberg uncertainty principle, as mentioned before. Therefore, the product of the uncertainties in the amplitude and phase quadratures cannot be smaller than a certain value, that we will study later. We can not know both the amplitude and phase with accuracy.

- Coherent states

Do you remember the annihilation operator â, known as amplitude operator too? I hope so! If not, don´t worry, read a few pages before and you got it.
Now that you know, let's explore the topic.

We define coherent states as the eigenstates of the annihilation operator.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

They exhibit a very smooth and stable behavior.

The eigenvalues of the annihilation operator are complex numbers, because the annihilation operator â is not Hermitian. Therefore, coherent states have complex wave functions. However, the expectation values of the quadrature operators (the â operator can be expressed in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$), provided by the complex wave functions of the coherent states, are real numbers and they can be calculated.

But it is important to note that these values are uncertain and subject to fluctuations due to the uncertainty principle in quantum mechanics. There is still an inherent quantum uncertainty associated with these values although the expectation values for a coherent state follow classical equations of motion.

Likewise, the complex wave functions of the coherent states are used in order to represent the quantum mechanical uncertainty and fluctuations inherent in all quantum states.

Coherent states are special types of quantum states that are used to describe the behavior of electromagnetic waves, such as light, in quantum mechanics. They are special in the sense that they behave very similarly to classical waves and their properties are very close to those of classical wave functions. This is why they can be known as classical states.

The main idea of coherent states is that they have the property of being as close as possible to a wave-like state while still satisfying the rules of quantum mechanics.

Any state that can be represented as a statistical mixture of coherent states is known as classical state. Nevertheless, any state that cannot be written as a coherent state ensemble, is called non-classical state. Thus, the distinction between classical and non-classical states in quantum optics is based on the degree to which the state can be understood in terms of classical concepts.

In addition to their wave-like behavior, coherent states are also special because they are the closest quantum states to a classical wave function. This is because the moduli squared of the coherent state wave function is proportional to the classical probability distribution for the position of the wave.

The behavior of a coherent state wave function is very similar to the behavior of a classical wave function.

Here's an alternative viewpoint on the coherent states, analyzing their properties.

Firstly, we need to remember the vacuum state. We have studied that it is a Fock state, because it presents a definite number of photons. In particular, zero.

However, it is often considered a coherent state due to its properties. Recently, we have concluded that $\hat{a}|0\rangle = 0$, because we rejected the idea that $\hat{a}^{\dagger}(\hat{a}|0\rangle) = 0$ when $\hat{a}|0\rangle \neq 0$.

$\hat{a}|0\rangle = 0$ follows the equation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ when α is equal to zero, and therefore, it is a coherent state.

Secondly, let us look at the properties of the coherent states.

1. **The coherent state can be expressed using the photon number eigenstates:**

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

As we mentioned before, the vacuum state it is both a coherent and Fock state. For this reason, and with the aid of an operator named displacement operator -$\mathrm{D}(\alpha)$, the coherent state can be expressed as a superposition of Fock states. The operator $\mathrm{D}(\alpha)$ where α is a complex number, is used to generate coherent states from the vacuum state.

It is defined as below:

$$\mathrm{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})$$

One important property of this operator is that it must be unitary. Thus, satisfy the equation:

$$\mathrm{D}^{\dagger}(\alpha)\ \mathrm{D}(\alpha) = \mathrm{I} = \mathrm{D}(\alpha)\mathrm{D}^{\dagger}(\alpha)$$

It can be proved by solving:

$$\mathrm{D}^{\dagger}(\alpha)\ \mathrm{D}(\alpha) = \exp(\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger})\exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})$$

What is more, we have said that the displacement operator describes coherent states from the vacuum state.

In mathematical terms, the displacement operator acts on the vacuum state as follows:

$$\mathrm{D}(\alpha)|0\rangle = |\alpha\rangle$$

When the $\mathrm{D}(\alpha)$ operator acts on the vacuum state, the result is a coherent state $|\alpha\rangle$. It is important to notice that this doesn't mean that coherent states are equal to vacuum states. Instead, they only share some quantum noise properties.

The noise properties in quantum mechanics refer to the uncertainties in the position and momentum of a quantum state. The position and momentum of a particle cannot both be zero.

The vacuum state, as the lowest energy state, has minimum uncertainties in its position and momentum, which are referred to as quantum noise. This quantum noise can be thought of as fluctuations in the field.

Similarly, coherent states, as they are a generalization of the vacuum state, also have similar quantum noise properties.

It can be demonstrated, not here but you can demonstrate it, that:

$$\mathrm{D}^{\dagger}(\alpha)\hat{a}\mathrm{D}(\alpha) = \hat{a} + \alpha$$

In words, the operator displaces the amplitude $\hat{a}$ by the complex number α.

Let´s see if it is true:

$$\hat{a}D(-\alpha)|\alpha\rangle = D^{\dagger}(-\alpha)D(-\alpha)\hat{a}D(-\alpha)|\alpha\rangle$$

The identity property ($D^{\dagger}(\alpha)\, D(\alpha) = I$) has been used.

$$D^{\dagger}(-\alpha)D(-\alpha)\hat{a}D(-\alpha)|\alpha\rangle = D(-\alpha)D^{\dagger}(-\alpha)\hat{a}D(-\alpha)|\alpha\rangle$$

As we know that "$D^{\dagger}(\alpha)D(\alpha) = D(\alpha)D^{\dagger}(\alpha)$".

$$D(-\alpha)D^{\dagger}(-\alpha)\hat{a}D(-\alpha)|\alpha\rangle = D(-\alpha)(\hat{a} - \alpha)|\alpha\rangle$$

Finally, applying: "$D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha$", we obtained the right-side equation. If we analyze it, we realize that:

$$(\hat{a} - \alpha)|\alpha\rangle = \hat{a}|\alpha\rangle - \alpha|\alpha\rangle$$

By definition, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, so:

$$(\hat{a} - \alpha)|\alpha\rangle = \alpha|\alpha\rangle - \alpha|\alpha\rangle = 0$$

That is to say,

$$\hat{a}D(-\alpha)|\alpha\rangle = |0\rangle\,;\;\; D(-\alpha)|0\rangle = |\alpha\rangle$$

It has been demonstrated then, that the operator displaces the amplitude $\hat{a}$ by the complex number α. At the same time, it has been confirmed that the operator acting on the vacuum state creates a coherent state. But also, that the displacement operator acting on the state $|\alpha\rangle$, produce the vacuum state. It shifts the state in phase space by the complex number $-\alpha$, from "α" to "$\alpha - \alpha$", that is, the vacuum.

2. The probability of finding the photon number n for the coherent state obeys the Poisson distribution:

$$p_n = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2)$$

Before studying the formula above, we are going to describe the relation between the vacuum and coherent states. We will start by calculating the quadrature wave functions.

We write the displacement operator in terms of the quadratures $\boldsymbol{q}$ and $\boldsymbol{p}$

$$D = \exp(ip_0\boldsymbol{q} - iq_0\boldsymbol{p})$$

Applying the Baker–Hausdorff formula; we can describe $D(\alpha)$ as

$$D(\alpha) = \exp(\frac{-i\, p_0 q_0}{2})\exp(i\, p_0 \boldsymbol{q})\exp(-iq_0 \boldsymbol{p})$$

Important to remark that $\exp(ip_0\boldsymbol{q})$ is equal to $\exp(ip_0 q)$ in the position representation.

The first factor, $\exp(\frac{-i\, p_0 q_0}{2})$, is a phase factor that depends on the real part of the displacement parameter α, which is equal to q_0. The second factor, $\exp(ip_0\boldsymbol{q})$, is an exponential factor that depends on the imaginary part of the displacement parameter α, which is equal to p_0. Finally, the third factor, $\exp(-iq_0\boldsymbol{p})$, is a unitary operator that shifts the wave function along the momentum axis by the amount q_0.

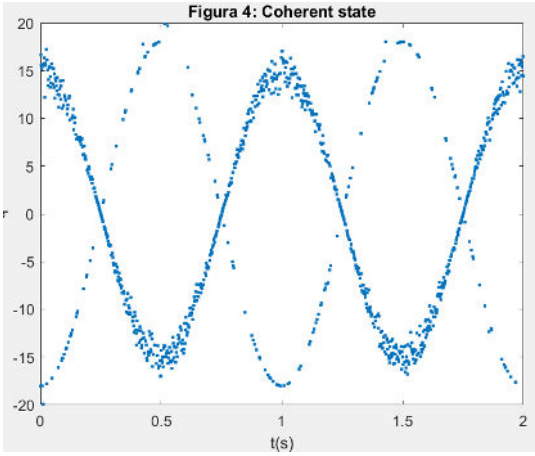
The formula is important because it highlights the relationship between the real and imaginary parts of the displacement parameter α and the effect of the displacement operator on the wave function.

This formula shows that the displacement operator acts on the position wave function by shifting it along the position axis, multiplying it by an exponential factor, and attaching a phase factor to it.

$$\psi_\alpha(q) = \psi_0(q - q_0)\exp(ip_0 q - \frac{i\, p_0 q_0}{2}) = \pi^{-\frac{1}{4}} \exp\left(-\frac{(q-q_0)^2}{2} + ip_0 q - \frac{i\, p_0 q_0}{2}\right)$$

The same idea for the momentum wave function:

$$\psi_\alpha(p) = \pi^{-\frac{1}{4}} \exp(-\frac{(p-p_0)^2}{2} - iq_0 p + \frac{i\, p_0 q_0}{2})$$



Position coherent state

The quadrature probability distributions $|\psi_\alpha(q)|^2$ and $|\psi_\alpha(q)|^2$ of coherent states are Gaussian distributions with the same width as the Gaussian curve for vacuum.

This distribution has the smallest uncertainty, just like the vacuum state.

The displacement operator $D(\alpha)$ can also be expressed in terms of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ as follows:

$$D(\alpha) = \exp(-\frac{|\alpha|^2}{2})\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})$$

Using the Baker–Hausdorff formula again, the above expression can be simplified to obtain the probability distribution $P_n = |\langle n|\alpha\rangle|^2$, which describes the probability of finding the harmonic oscillator in the n-th energy state when it is in a coherent state.

Let´s evaluate the inner product of the coherent state $|\alpha\rangle$ with the Fock state $|n\rangle$, which is given by:

$$\langle n|\alpha\rangle = \langle n|D(\alpha)|0\rangle = \langle n|\exp(-\frac{|\alpha|^2}{2})\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})|0\rangle$$

The only thing we did is just express $|\alpha\rangle$ in terms of the displacement operator, applying its definition calculated before.

After some algebraic manipulation, one can obtain the expression for the probability distribution, by taking the square of the magnitude of $\langle n|\alpha\rangle$:

$$P_n = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2)$$

This equation is known as the Poisson distribution, and it describes the probability of finding n photons in a state described by the complex amplitude α.

3. The mean and variance of the photon number for a coherent state $|\alpha\rangle$ are:

$$\langle \boldsymbol{n} \rangle = |\alpha|^2$$

$$\langle \Delta \boldsymbol{n}^2 \rangle = |\alpha|^2$$

The mean photon number for a coherent state $|\alpha\rangle$ can be calculated using the following formula:

$$\langle \boldsymbol{n} \rangle = \sum_n nP_n = |\alpha|^2$$

where P_n is the probability of finding n photons in the state, given by the Poisson distribution analyzed in the previous paper.

Furthermore, the variance can be calculated using the mean photon number:

$$\langle \Delta \boldsymbol{n}^2 \rangle = \langle \boldsymbol{n}^2 \rangle - \langle \boldsymbol{n} \rangle^2$$

Substituting the necessary values into the equation, the variance ends up being equal to the mean:

$$\langle \Delta \boldsymbol{n}^2 \rangle = |\alpha|^2 = \langle \boldsymbol{n} \rangle$$

This result shows that the variance of the photon number is equal to the mean photon number, which implies that the photon number is not very uncertain in a coherent state. This property is what makes coherent states an important class of states in quantum optics, as they provide a way to describe light that is very close to a classical light wave. This explains why the coherent states or vacuum states are often used as a reference, because they have relatively small fluctuations.

The mean energy of a coherent state $|\alpha\rangle$ can also be calculated, as the expectation value of the Hamiltonian operator $\hat{H}$ over the state. The Hamiltonian operator for the harmonic oscillator, which is commonly used to describe the energy of a coherent state, is given by:

$$\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, respectively. I remember it, but you should already know it ʘ_ʘ.

The expectation value of the Hamiltonian operator over the state $|\alpha\rangle$ can be calculated as:

$$\langle \alpha | H | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} + \frac{1}{2} | \alpha \rangle = |\alpha|^2 + \frac{1}{2}$$

So, the mean energy of a coherent state $|\alpha\rangle$ can be expressed as the sum of the quantum and classical energies, where the quantum energy is proportional to the magnitude of the displacement $|\alpha|^2$ and the classical energy is equal to $\frac{1}{2}$.

The mean energy of a vacuum state, which is a special case of the coherent state with $\alpha = 0$, is equal to $\frac{1}{2}$, which is the minimum possible energy for the harmonic oscillator.

4. Not exactly orthogonal:

The inner product between two coherent states $|\alpha'\rangle$ and $|\alpha\rangle$ is given by:

$$\langle \alpha' | \alpha \rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(-\frac{|\alpha'|^2}{2}\right) \exp(\alpha'^* \alpha)$$

Here, $|\alpha|^2$ and $|\alpha'|^2$ represent the average photon number of the coherent states, and $\alpha'^*\alpha$ is the complex conjugate of α' multiplied by α. The exponential term represents the overlap between the two coherent states in phase space.

So, the square of the magnitude of the inner product is:

$$|\langle\alpha'|\alpha\rangle|^2 = \exp(-|\alpha - \alpha'|^2)$$

Coherent states are not exactly orthogonal to each other. As we have seen, the inner product between two different coherent states does not vanish, it is not equal to zero. This is because coherent states are not eigenstates of a Hermitian operator, therefore, they don't satisfy the criterion for being an orthogonal set of states.

However, when the difference between the amplitudes α and α' becomes significantly larger than the quadrature noise level of the vacuum, it can be considered that the two states are largely orthogonal and their overlap is negligible. In other words, when the amplitudes of two coherent states differ sufficiently, their inner product becomes very small and can be considered as approximately orthogonal.

5. **Coherent states are complete:**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\alpha\rangle\langle\alpha|\frac{dq_0p_0}{2\pi} = 1$$

They form a complete basis for the Hilbert space of a quantum system. Hence, any quantum state can be represented as a superposition of coherent states.

- Thermal states

To finish with, we are going to talk about thermal states.

Light itself does not have the properties to reach an equilibrium state. However, when it comes into contact with a material, it can transfer its energy to the material and cause it to heat up, eventually reaching an equilibrium state with its surroundings. Light can reach an equilibrium state, but it cannot do so on its own.

In order to describe thermal states, we need to know that a thermal state is a state of light such that the light is in a superposition of many Fock states.

$$\boldsymbol{\rho} = \sum_n \rho_n |n\rangle\langle n|$$

We have already study that Fock states are characterized as the eigenstates of the photon number operator, $\boldsymbol{n}$. For this reason, when light interacts with a hot material, as the electromagnetic field exchanges photons with the material, the energy and the number of photons in the light fluctuates. These fluctuations result in thermal light.

The reason of the random and unpredictable changes in the energy and photon number of the electromagnetic field can be explained by the random motion of the thermal particles.

The exchange of photons between the light and the material causes the light to have a range of photon numbers, rather than a definite number as in a Fock state. Hence, the continuous absorption and emission of photons between light and the hot material results in a state of maximum entropy - remember that the entropy is a measure of the amount of uncertainty or disorder of the system.

As the system is constantly fluctuating and the number of photons and their energies are not well-defined, the information about the system is maximally uncertain. Therefore, the exchange of photons in thermal light is random and incoherent.

The entropy is expressed as:

$$S = -k\,\mathrm{tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\}\ ;\ k \equiv \text{Boltzmann constant}$$

As the trace is:

$$\mathrm{tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\} = \sum_n \rho_n \ln \rho_n$$

The entropy can be rewritten,

$$S = -k\,\mathrm{tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\} = -k \sum_n \rho_n \ln \rho_n$$

When thermal equilibrium is reached, entropy adopts the value:

$$S = -k \ln Z + \frac{E}{T}$$

where Z is the partition function, E is the average energy of the system, and T is the temperature of the system.

$$E = \sum_n \rho_n E_n$$

$$Z = \sum_n \exp\left(\frac{-k\,E_n}{T}\right)$$

The statistical sum (also known as partition function) takes into account the contributions of all possible states of the system.

The thermodynamic temperature, T, can be expressed as the derivate of the entropy with respect to the energy, E:

$$T = \frac{\partial E_n}{\partial S}$$

where the derivative is taken at constant volume.

According to the second law of thermodynamics, the entropy of a closed system increases over time as heat is transferred from hotter to colder regions. The

temperature of a system represents the rate at which its thermal energy changes in relation to its entropy.

Going a little bit further, we know that a light mode of frequency ω can be treated as a simple harmonic oscillator. Therefore, we will focus on the thermal state of an electromagnetic oscillator in a light mode, and study the partition function, the density matrix, and the average photon number.

For a single light mode, the partition function is given by:

$$Z = \frac{1}{1 - e^{-\beta}}$$

It's noteworthy to mention that:

$$\beta = \frac{\hbar\omega}{kT}$$

What is more, the density matrix of a thermal state, that represents the probabilities of occurrence of all possible energy states, is described in symbols as:

$$Z = (1 - e^{-\beta}) \sum_{n=0}^{\infty} e^{-n\beta} |n\rangle\langle n|$$

Finally, the average photon number, represented by $\bar{n}$, is a measure of the average number of photons in a light mode at a given temperature T. The Planck spectrum of a harmonic oscillator in thermal equilibrium gives the relationship between the average photon number and the temperature of the system. The formula for the average photon number is given by:

$$\bar{n} = \frac{1}{e^{\beta} - 1}$$

The average photon number increases with temperature, as we can realize in the formula above, as β is indirectly proportional to T. It reaffirms the idea that more energy is required to excite the harmonic oscillator as the temperature increases.

Moreover, when the entire electromagnetic field is in a thermal state, in other words, it is in a state of equilibrium with a temperature, the energy density of the field can be calculated.

The energy density per unit volume and frequency is obtained by adding up the energies of all the modes ($\hbar\omega\bar{n}$) that have the same frequency ω and dividing the sum by the total volume:

$$\varrho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\beta} - 1}$$

where c is the speed of light.

This formula gives the energy density of a single light mode in a thermal state at temperature T and frequency ω, and $\hbar$ is known as the Plank´s radiation formula.

- Uncertainty and squeezing

The position and momentum of a particle cannot be simultaneously determined with arbitrary accuracy due to the Heisenberg Uncertainty Principle. This principle states that the product of the uncertainties in position and momentum cannot be smaller than a constant value, half of Planck's constant.

Therefore, in any quantum state, there will always be some level of uncertainty in both position and momentum. However, in minimum uncertainty states, the uncertainties in position and momentum are optimized such that their product is as close to Planck's constant as possible.

To show an example of a minimum uncertainty state in position space, let´s start with $|\psi\rangle$, a possible candidate state.

We calculate the average complex amplitude of the state $|\psi\rangle$, which is equal to the expectation value:

$$\langle\psi|\hat{a}|\psi\rangle = \alpha$$

In words, the overall amplitude of the state $|\psi\rangle$ is represented by α.

For simplicity, the overall amplitude must be removed. This allows us to focus solely on the fluctuations in the position and momentum quadratures, rather than the overall amplitude of the state. The displacement operator shifts the origin of the state in phase space, so that the fluctuations can be analyzed without being influenced by the overall amplitude.

As said, let´s apply the displacement operator, which will generate a new state with the same amount of quadrature noise as the initial state:

$$|\phi\rangle = D(-\alpha)|\psi\rangle$$

After applying the displacement operator $D(-\alpha)$, the average value of the complex amplitude, represented by $\langle\phi|\hat{a}|\phi\rangle$, becomes 0, as the overall amplitude has been removed. The state $|\phi\rangle$ is centered at the origin in phase space, making it easier to analyze the fluctuations.

From the expression

$$\delta = \left|\frac{q\phi}{2\Delta^2 q} + \frac{\partial\phi}{\partial q}\right|^2$$

Which represents the combined uncertainty in the position and momentum of the state $|\phi\rangle$.

The first term represents the position fluctuations in the state $|\phi\rangle$, while the second one represents the fluctuations in the momentum.
The absolute value is taken because the values of the position and momentum cannot be simultaneously known with complete certainty, so their fluctuations must always be non-negative. Essentially, greater than or equal to cero.

If we integrate δ and make some assumptions, we will obtain:

$$\int \delta \, dq = \frac{-1}{4\Delta^2 q} + \Delta^2 p \geq 0$$

This result is directly related to Heisenberg's uncertainty relation. The uncertainty relation states that for any quantum state, the product of the uncertainties in the position and momentum observables must be greater than or equal to one-half.

Mathematically, the Heisenberg's uncertainty relation with $\hbar = 1$, for

$$\Delta q = \sqrt{\Delta^2 q}$$

$$\Delta p = \sqrt{\Delta^2 p}$$

Is described as:

$$\Delta q \Delta p \geq \frac{1}{2}$$

Finally, we can represent a minimum uncertainty state.

The mathematical trick of integrating the expression:

$$\delta = \left|\frac{q\phi}{2\Delta^2 q} + \frac{\partial \phi}{\partial q}\right|^2$$

allows us to quickly identify the minimum uncertainty states, because the equality sign in the relation $\frac{-1}{4\Delta^2 q} + \Delta^2 p \geq 0$ holds only when:

$$\frac{1}{2}\frac{q}{\Delta^2 q}\phi + \frac{\partial \phi}{\partial q} = 0.$$

Therefore,

$$\phi(q) = (2\pi\Delta^2 q)^{-\frac{1}{4}} \exp\left(-\frac{q^2}{4\Delta^2 q}\right)$$

It is described by a Gaussian distribution in both position and momentum space.

So, we have demonstrated that the product of the uncertainties of position and momentum must be greater than or equal to $\frac{1}{2}$. Hence, in the case of a minimum-uncertainty state, the uncertainties are equal to $\frac{1}{\sqrt{2}}$.

However, it is possible to squeeze one of the uncertainties (position or momentum) below this value, but the uncertainty in the other quadrature increases.

The squeezing parameter ζ is used to describe the deviation of the variances from their vacuum values. Positive values of ζ correspond to squeezing in the position quadrature, while negative values correspond to squeezing in the momentum quadrature.

$$\Delta^2 \mathrm{q} = \frac{1}{2} e^{-2\zeta}$$

The equation above represents the relationship between the variance of position (the left side of the formula) and the squeezing parameter. According to the equation, the variance of position decreases as the squeezing parameter increases.

In the same manner,

$$\Delta^2 \mathrm{p} = \frac{1}{2} e^{+2\zeta}$$

We can write the squeezing operator as

$$\hat{\mathrm{S}}(\zeta) = \exp\left(\frac{\zeta}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2)\right)$$

The squeezing operator is a unitary operator that is used to generate squeezed states in quantum mechanics. Squeezed states are states where the uncertainty in one of the quadratures has been reduced below the vacuum level, as mentioned before, increasing the uncertainty in the other quadrature.

The squeezing operator acts on the vacuum state to generate a squeezed vacuum state, which can be written as:

$$|\phi\rangle = \hat{\mathrm{S}}(\zeta)|0\rangle$$

Moreover, states that have the minimum possible uncertainty in their position and momentum are referred to as "displaced squeezed vacua". These states can be represented by a wave function, $|\psi\rangle$, that is formed by the combination of a displacement operator, $\mathrm{D}(\alpha)$, and a squeezing operator, $\hat{\mathrm{S}}(\zeta)$.

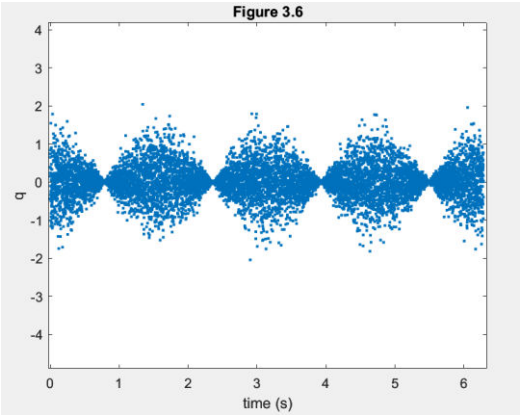
In mathematical terms,

$$|\psi\rangle = \mathrm{D}(\alpha)\hat{\mathrm{S}}(\zeta)|0\rangle$$

The displacement operator moves the vacuum state, $|0\rangle$, to a new state that has a non-zero expectation value for position and momentum, while the squeezing operator reduces the uncertainty in either position or momentum while maintaining the total uncertainty constant.

Consequently, the position wave function is:

$$\psi(q) = \pi^{-\frac{1}{4}} e^{\frac{\zeta}{2}} \exp(-e^{+2\zeta} \frac{(q-q_0)^2}{2} + ip_0 q - \frac{ip_0 q_0}{2})$$



Quadrature noise of a squeezed vacuum

The production of a squeezed vacuum requires a pump, which means that the resulting state carries energy. This energy is used to produce a nonlinear interaction and therefore, is what allows the squeezing operator to produce a squeezed vacuum. The energy provided is the necessary for the squeezing operator to change the fluctuations in the vacuum state and be able to generate the state.

It is important to remark that squeezed vacuum states are not pure vacuum states. The latter, refers to a state with no energy or particles present. The presence of energy in the pump field means that the vacuum fluctuations have been altered and are no longer in a pure and undisturbed state.

The energy used to generate the states must be carefully controlled in order to produce the desired degree of squeezing. If the energy of the pump field is too high, it can result in the production of excess noise in the squeezed vacuum state, which reduces the quality of the squeezing.

The squeezing operator amplifies the vacuum fluctuations that are in-phase with the pump and deamplifies the fluctuations that are out-of-phase with the pump.

The squeezing operator has effect on the quadratures when apply to them:

$$\hat{S}^{\dagger}(\zeta)\, \boldsymbol{q}\, \hat{S}(\zeta) = \boldsymbol{q}\, e^{-\zeta}$$

$$\hat{S}^{\dagger}(\zeta)\boldsymbol{p}\hat{S}(\zeta) = \boldsymbol{p}\,e^{+\zeta}$$

The squeezing operator changes the properties of the quadratures when it is applied to them. It does this by scaling the eigenfunctions of the quadratures.

The expressions can be rewritten as:

$$\hat{S}^{\dagger}(\zeta)\hat{a}\hat{S}(\zeta) = \hat{a}\cosh(\zeta) - \hat{a}^{\dagger}\sinh(\zeta)$$

Taking this into consideration, the energy

$$\langle\psi|\hat{E}|\psi\rangle = |\alpha|^2 + \frac{1}{2} + \sinh^2(\zeta)$$

The first factor is the energy associated with the coherent amplitude, which is given by the square of the magnitude of α. On the other hand, the second factor is the vacuum energy, which is always present and equal to $\frac{1}{2}$. Finally, the last factor is the fluctuation energy, which is unique to squeezed states and arises from the enhanced fluctuations in the stretched component.

The total energy of a squeezed vacuum state is a combination of both the squeezed and stretched quadratures, even though it is just the squeezed vacuum. This means that the energy of a squeezed vacuum state is not equal to zero, even though it is not a pure vacuum state.

The photon number statistics of a squeezed vacuum can be calculated by using the number operator, which measures the number of photons in the state:

$$p_n = |\langle n|\phi\rangle|^2 = |\langle n|\hat{S}(\zeta)|0\rangle|^2$$

Where $\langle n|\hat{S}(\zeta)|0\rangle$ is defined as:

$$\langle n|\hat{S}(\zeta)|0\rangle = \int_{-\infty}^{\infty} \psi_n(q)e^{\frac{\zeta}{2}}\psi_0(e^{\zeta}q)\,dq$$

It bears mentioning that the squeezed vacuum states only contains photon pairs. This is due to the mirror symmetry of squeezing and because it is created through a process called parametric process, which is described by the Hamiltonian.

On one hand, they are symmetrical in nature, where the sign of the quadrature amplitude q does not affect their wave function as it remains even ($\psi(q) = \psi(-q)$).

This property can be seen in the wave functions of the Fock states, which are even for even photon numbers and odd for odd photon numbers. This results in the scalar product integral "$\langle n|\hat{S}(\zeta)|0\rangle = \int_{-\infty}^{\infty} \psi_n(q)e^{\frac{\zeta}{2}}\psi_0(e^{\zeta}q)\,dq$" to vanish for odd photon numbers, where

$$p_{2m+1} = 0\ (m = 0, 1, 2, \ldots)$$

On the other hand, squeezed vacuum states are created through a process known as parametric amplification, where the Hamiltonian of the system changes over time due to the interaction between light and a non-linear material.

When light interacts with a material, it can change its properties and produce energy. The change in the light's properties and the production of energy is described by the Hamiltonian, which acts as an energy operator. The result of this interaction is the creation of a type of non-classical light known as a squeezed vacuum state.

There are pump photons used to drive the interaction between the light and the material. When each of them interacts with the material, it is converted into two signal photons of half the pump frequency. Therefore, only photon pairs are produced.

The probability to find a photon pair is described as:

$$p_{2m} = \binom{2m}{m} \frac{1}{\cosh(\zeta)} \left(\frac{\tanh(\zeta)}{2}\right)^{2m} \quad (m = 0, 1, 2, \ldots)$$

It can be noticed that each pair is independent of the others. In other words, the change in one photon pair does not affect the other photon pairs.

However, the photons within each pair are correlated, so their properties are related to each other. If there is a change in one of the photons, the other is also affected.

The last important idea to mention is that photons are produced in pairs, as we know. Nevertheless, when trying to measure the photons, due to the limitations of the detectors used, sometimes only one photon of a pair may be observed, even though the photons were generated in pairs.

3. BIBLIOGRAPHY

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